B. Inclusion of Surface Force

Assume an arbitrary force that acts at the surface of the droplet and causes a distortion of the droplet static shape. We assume that the interaction between the drop oscillation and the external field is negligible. In general, the force can be expanded in terms of spherical harmonics as

$$P_{\text{ext}} = \sum C_{l,m}^{n} \left(\frac{R + X + Z}{R} \right)^{n} Y_{lm}(\theta, \chi) \tag{22}$$

This is a Taylor-series expansion of an arbitrary function about the origin in spherical coordinates, and P_{ext} is assumed to be sufficiently continuous. The surface condition

$$\delta p = \gamma (1/R_1 + 1/R_2) + P_{\text{ext}} \tag{23}$$

can now be applied. The time-dependent terms in the complete surface condition give after simplification

$$(\omega_{lm})^2 = \frac{\gamma l \alpha_4}{\rho R^3 \alpha_2} \Bigg(-\alpha_1 + l(l-2)\alpha_5 + l\alpha_6 + 2\sum_{u=1}^{\infty}\sum_{v=-u}^{u} \frac{X_{uv} I_{um}^{(l)}}{R} \{\alpha_1 - [l(l-2) + u(u-2)]\alpha_5 + (l+u)\alpha_6\} + 3\sum_{u=1}^{\infty}\sum_{v=-u}^{u} \frac{X_{uv} I_{um}^{(l)}}{R^2} \sum_{u=1}^{\infty}\sum_{v=-u}^{u} X_{uv} I_{um}^{(l)} \{-\alpha_1 + [l(l-2) + 2u(u-2)]\alpha_5 + [l+2u]\alpha_6\} + \sum \frac{n}{R} C_{uv}^{n} I_{um}^{(l)} \Bigg)$$
$$\times \Bigg(1 - \Bigg\{ \sum_{u=1}^{\infty}\sum_{v=-u}^{u} X_{uv} I_{um}^{(l)} + [(l-1)/R] \left[\sum_{u=1}^{\infty}\sum_{v=-u}^{u} X_{uv} I_{um}^{(l)} \right]^2 \Bigg\} \Bigg/ \Bigg\{ R + l \sum_{u=1}^{\infty}\sum_{v=-u}^{u} X_{uv} I_{um}^{(l)} + [l(l-1)/2R] \left[\sum_{u=1}^{\infty}\sum_{v=-u}^{u} X_{uv} I_{um}^{(l)} \right]^2 \Bigg\} \Bigg) \tag{24}$$

IV. Conclusions

In this work the oscillations of an oblate spheroid shape inviscid droplet subjected to external forces have been considered. The analysis developed can be extended to consider different static deformation shapes. We have explained the frequency splitting frequently observed in experiments and are able to interpret these split spectra more accurately in thermophysical property measurements. Comparisons between experiment data and analytical predictions concerning the frequency splitting of deformed levitating droplets are instructive for evaluating the state of knowledge of the forces acting on a levitated droplet.

Acknowledgments

The authors are grateful for the financial support by National Science Foundation Grant CTS 9312379 and by Texas Advanced Technology Program Grant 003604-056.

References

[1]Rayleigh, J. W. S., "The Equilibrium of Revolving Liquid Under Capillary Force," *Philosophical Magazine*, Vol. 28, No. 1, 1914, pp. 161–170.

[2]Lamb, H., *Hydrodynamics*, 6th ed., Dover, New York, 1945, p. 473.

[3]Bayazitoglu, Y., and Suryanarayana, P. V. R., "Dynamics of Viscous Oscillating Viscous Droplets Immersed in Viscous Media," *Acta Mechanica*, Vol. 95, No. 1–4, 1992, pp. 167–183.

[4]Miller, C. A., and Scriven, L. E., "The Oscillations of a Fluid Droplet Immersed in Another Liquid," *Journal of Fluid Mechanics*, Vol. 32, No. 3, 1968, pp. 417–435.

[5]Prosperetti, A., "Normal-Mode Analysis for the Oscillations of a Viscous Liquid Drop in a Immiscible Liquid," *Journal de Mecanique*, Vol. 19, No. 1, 1980, pp. 149–182.

[6]Basaran, O., "Nonlinear Oscillations of Viscous Liquid Drops," *Journal of Fluid Mechanics*, Vol. 241, Aug. 1992, pp. 169–198.

[7]Marston, P. L., "Shape Oscillations and Static Deformation of Drops and Bubbles Driven by Modulated Radiation Stresses-Theory," *Journal of the Acoustical Society of America*, Vol. 67, No. 1, 1980, pp. 15–26.

[8]Cummings, D. L., and Blackburn, D. A., "Oscillations of Magnetically Levitated Aspherical Droplets," *Journal of Fluid Mechanics*, Vol. 224, March 1991, pp. 395–413.

[9]Suryanarayana, P. V. R., and Bayazitoglu, Y., "Effect of Static Deformation and External Forces on the Oscillations of Levitated Droplets," *Physics of Fluids A*, Vol. 3, No. 5, 1991, pp. 967–977.

[10]Bayazitoglu, Y., and Mitchell, G. F., "Experiments in Acoustic Levitation: Surface Tension Measurements of Deformed Droplets," *Journal of Thermophysics and Heat Transfer*, Vol. 9, No. 4, 1995, pp. 694–701.

[11]Mitchell, G. F., Bayazitoglu, Y., and Shampine, R., "Acoustically Levitated Droplet Viscosity Measurement," *Journal of Thermophysics and Heat Transfer*, Vol. 12, No. 4, 1998, pp. 599–601.

Interaction of Radiation with Natural Convection

C. K. Krishnaprakas* and K. Badari Narayana†
ISRO Satellite Centre, Bangalore 560 017, India
and
Pradip Dutta‡
Indian Institute of Science, Bangalore 560 012, India

Nomenclature

f	= dimensionless stream function, $(Gr/4)^{-1/4}\psi(x, y)/(4\nu)$
Gr	= Grashof number, $g\lambda\lvert T_w - T_\infty\rvert x^3/\nu^2$
g	= acceleration caused by gravity, m/s², dimensionless intensity function
$I_b(T)$	= blackbody radiation intensity, W/m²-sr
$I^+(\upsilon, \mu, \xi)$	= radiation intensity for positive values of μ, W/m²-sr
$I^-(\upsilon, -\mu, \xi)$	= radiation intensity for negative values of μ, W/m²-sr
k	= thermal conductivity, W/m-K
N_c	= conduction-radiation number = $k\beta/4\sigma T_w^3$
Pr	= Prandtl number, ν/α
$p(\mu, \mu')$	= slab-scattering phase function
$p(\mu_p)$	= single-scattering phase function
Q^r	= dimensionless radiative heat flux, $q^r/4\sigma T_w^4$
q^r	= radiative heat flux, W/m²
q_w^t	= total heat flux from wall, W/m²
T	= temperature, K
T_w	= hot-plate temperature, K
T_∞	= ambient temperature, K
u	= x-direction velocity component, m/s = $\partial\psi/\partial y$
v	= y-direction velocity component, m/s = $-\partial\psi/\partial x$
x, y	= coordinates respectively in the parallel and perpendicular directions of the plate, m
α	= thermal diffusivity, m²/s

Received 26 May 1998; revision received 11 January 1999; accepted for publication 15 February 1999.

*Engineer; currently Research Scholar, Indian Institute of Science, Bangalore 560 012, India.

†Head, Thermal Design Section, Thermal Systems Group.

‡Assistant Professor, Mechanical Engineering Department.

β = extinction coefficient, $\kappa + \gamma$, m^{-1}
γ = scattering coefficient, m^{-1}
ε = emittance of plate surface
η = dimensionless distance, $(Gr/4)^{1/4}(y/x)$
θ = dimensionless temperature, T/T_w
θ_∞ = T_∞/T_w
κ = absorption coefficient, m^{-1}
λ = coefficient of thermal expansion, K^{-1}
μ = $\cos\chi$
ν = kinematic viscosity, m^2/s
ξ = dimensionless axial distance (Bouguer number), $\beta x/(Gr/4)^{1/4}$
ρ = density, kg/m^3
ρ^d = diffuse reflectance of plate surface, $1-\varepsilon$-ρ^s
ρ^s = specular reflectance of plate surface, $1-\varepsilon$-ρ^d
σ = Stefan–Boltzmann constant, W/m^2-K^4
υ = optical depth of the medium at y defined by $d\upsilon = \beta\, dy$
χ = polar angle, angle between the radiation vector and the y axis
ψ = stream function
ω = scattering albedo, γ/β

Introduction

THE effect of radiation interaction is more significant for natural convection rather than for forced convection. The interaction of radiation with laminar natural convection from a vertical plate was first modeled by Cess[1] for an absorbing and emitting fluid for the optically thick case employing a singular perturbation technique. Arpaci[2] included both optically thin and thick cases for an absorbing and emitting medium and solved the problem using an approximate integral technique. Later, Cheng and Ozisik[3] extended the analysis to include isotropic scattering and employed an exact treatment (normal mode expansion technique) for the solution of the radiation part. The objective of the present Note is to investigate the effects of the degree of anisotropy in radiation scattering on the heat flux when combined laminar natural convection and radiation takes place from a heated vertical plate to an absorbing, emitting, and scattering fluid.

Mathematical Model

The governing equations are those of the conservation of mass, momentum, and energy equations for a vertical plate maintained at a uniform temperature T_w immersed in a radiatively absorbing, emitting, and scattering gray fluid at a temperature T_∞ outside the boundary layer. The stream function and vorticity formulation is used. The equations are expressed in a dimensionless form as[3]

$$\frac{\partial^3 f}{\partial\eta^3} + 3f\frac{\partial^2 f}{\partial f^2} - 2\left(\frac{\partial f}{\partial\eta}\right)^2 + \frac{\theta-\theta_\infty}{1-\theta_\infty} = \xi\left(\frac{\partial^2 f}{\partial\xi\partial\eta}\frac{\partial f}{\partial\eta} - \frac{\partial f}{\partial\xi}\frac{\partial^2 f}{\partial\eta^2}\right) \tag{1}$$

$$\frac{1}{Pr}\frac{\partial^2\theta}{\partial\eta^2} + 3f\frac{\partial f}{\partial\theta} = \xi\left(\frac{\partial f}{\partial\eta}\frac{\partial\theta}{\partial\xi} - \frac{\partial f}{\partial\xi}\frac{\partial\theta}{\partial\eta}\right) + \frac{\xi}{N_c Pr}\frac{\partial Q^r}{\partial\eta} \tag{2}$$

The boundary conditions are

$$f = \frac{\partial f}{\partial\eta} = 0 \quad \text{at} \quad \eta = 0 \tag{3a}$$

$$\frac{\partial f}{\partial\eta} = 0 \quad \text{at} \quad \eta = \infty \tag{3b}$$

$$\theta = 1 \quad \text{at} \quad \eta = 0 \tag{3c}$$

$$\theta = \theta_\infty \quad \text{at} \quad \eta = \infty \tag{3d}$$

$$f = f_0 \quad \text{and} \quad \theta = \theta_0 \quad \text{at} \quad \xi = 0 \tag{3e}$$

where f_0 and θ_0 are the solutions to the problem for the pure natural convection when there is no radiation.

Radiative Flux Calculation

The radiative transfer equation describing the intensity variation for an absorbing, emitting, scattering, gray, semi-infinite, plane-parallel medium may be written as[4]

$$\mu\frac{\partial}{\partial\upsilon}I(\upsilon,\mu,\xi) + I(\upsilon,\mu,\xi) = (1-\omega)I_b[T(\upsilon,\xi)] + \frac{\omega}{2}\int_{-1}^{1} p(\mu,\mu')I(\upsilon,\mu',\xi)\,d\mu' \tag{4}$$

The boundary conditions are

$$I(0,\mu,\xi) = \varepsilon I_b(T_w) + 2\rho^d\int_0^1 I(0,-\mu,\xi)\mu\,d\mu + \rho^s I(0,-\mu,\xi), \qquad \mu > 0 \tag{5}$$

$$I(\upsilon=\infty,-\mu,\xi) = I(\upsilon=\infty,\mu,\xi) \tag{6}$$

The optical depth variable υ is defined as $d\upsilon = \beta\, d\eta$. The relation between υ, η, and ξ is given as

$$\upsilon = \xi\eta \tag{7}$$

Equation (6) means that at υ(or η) $=\infty$ the radiative flux vanishes. For numerical computations ∞ is represented by $\upsilon_{max} = \xi\eta_{max}$, where η_{max} is a sufficiently large value representing the edge of the boundary layer. In the subsequent paragraphs the variable ξ is omitted while writing $I(\upsilon,\mu,\xi)$, for the sake of brevity, because we are only concerned with the evaluation of intensity at a particular ξ station.

Discrete ordinates method (DOM)[5] is used to transform the integrodifferential equation (4) into a system of coupled linear ordinary differential equations (ODEs). Use of DOM for planar radiating flow is discussed by Cheng and Lui.[6] We follow the implementation of DOM employed by Love and Grosh.[7] For convenience $I(\upsilon,\mu)$ is separated into a forward component $I(\upsilon,\mu)$, $\mu\in(0,1)$ and a backward component $I(\upsilon,-\mu)$, $\mu\in(0,1)$, and an m-point Gauss–Legendre numerical quadrature rule is applied to evaluate the scattering integral in Eq. (4) for each component:

$$\mu_i\frac{dI(\upsilon,\mu_i)}{d\upsilon} + I(\upsilon,\mu_i) = (1-\omega)I_b[T(\upsilon)] + \frac{\omega}{2}\sum_{j=1}^{m} w_j[p(\mu_i,\mu_j)I(\upsilon,\mu_j) + p(\mu_i,-\mu_j)I(\upsilon,-\mu_j)] \tag{8}$$

$$-\mu_i\frac{dI(\upsilon,-\mu_i)}{d\upsilon} + I(\upsilon,-\mu_i) = (1-\omega)I_b[T(\upsilon)] + \frac{\omega}{2}\sum_{j=1}^{m} w_j[p(\mu_i,-\mu_j)I(\upsilon,\mu_j) + p(\mu_i,\mu_j)I(\upsilon,-\mu_j)] \tag{9}$$

where w_i, μ_i, $i = 1, m$ are the weights and abscissas of the quadrature rule. The boundary condition equations with the quadrature rule applied are written as

$$I(0,\mu_i) = \varepsilon I_b(T_w) + 2\rho^d\sum_{j=1}^{m} w_j\mu_j I(0,-\mu_j) + \rho^s I(0,-\mu_j) \tag{10}$$

$$I(\upsilon_{max},-\mu_i) = I(\upsilon_{max},\mu_i) \tag{11}$$

Equations (8–11) constitute a two-point boundary value problem in ODEs. These equations are solved by a shooting method in conjunction with the Crank–Nicolson (CN) method as core integrator.

The iterative scheme is tailored with the Ng-acceleration scheme described by Auer[8] to speed up convergence.

Equation (2) contains the divergence of the radiative flux term as the last term. This is evaluated as[5]

$$\frac{\partial Q^r}{\partial \eta} = (1-\omega)\xi[\theta^4(\eta) - g(\eta)] \tag{12}$$

where $g(\eta)$ is the dimensionless incident radiation function evaluated from the radiation intensity distribution $I[\upsilon(\eta), \mu]$ as[5]

$$g(\eta) = \frac{\pi}{2\sigma T_w^4}\int_{-1}^{1} I[\upsilon(\eta), \mu]\,\mathrm{d}\mu = \frac{\pi}{2\sigma T_w^4}\sum_{j=1}^{m} w_j\{I[\upsilon(\eta), \mu_j] + I[\upsilon(\eta), -\mu_j]\} \tag{13}$$

The dimensionless radiative heat flux Q^r may now be obtained as

$$Q^r = \frac{q^r}{4\sigma T_w^4} = \frac{\pi}{2\sigma T_w^4}\int_{-1}^{1} I(\upsilon, \mu)\mu\,\mathrm{d}\mu = \frac{\pi}{2\sigma T_w^4}\sum_{j=1}^{m} w_j\mu_j[I(\upsilon, \mu_j) - I(\upsilon, -\mu_j)] \tag{14}$$

The total heat flux at the wall is the sum of the convective and radiative contributions and is given as

$$q_w^t = \left[-k\frac{\partial T}{\partial y} + q^r\right]_{y=0} \tag{15}$$

A local Nusselt number may now be defined to represent the total heat flux from the wall to the fluid as

$$Nu = \frac{q_w^t x}{k(T_w - T_\infty)} \tag{16}$$

Substitution of Eq. (15) into Eq. (16) results in

$$\frac{Nu}{(Gr/4)^{\frac{1}{4}}} = \frac{1}{1-\theta_\infty}\left[-\frac{\partial\theta}{\partial\eta} + \frac{\xi}{N_c}Q^r\right]_{\eta=0} \tag{17}$$

Numerical Scheme

Solution of the partial differential system, Eqs. (1) and (2), along with the boundary conditions (3a–3e), obtains the velocity and temperature profiles within the boundary layer. Finite differencing in the ξ variable converts the partial differential system into a two-point boundary value problem in ODEs. The resulting problem is solved using the shooting method. The numerical procedure starts with an initial approximation to the profile of temperature at the ξ_k station; using this the radiative divergence term is calculated, and the ODEs are integrated, using initial guesses of the $\mathrm{d}^2 f/\mathrm{d}\eta^2$, $\mathrm{d}\theta/\mathrm{d}\eta$ at $\eta = 0$ to $\eta = \eta_{\max}$, where $\eta_{\max}$ is a sufficiently large value taken to represent the edge of the boundary layer. At $\eta = \eta_{\max}$ the solution should satisfy the boundary conditions (3b) and (3d). A residual function $F(x)$ may now be defined as

$$F(x) = \boldsymbol{r}(x)^T\boldsymbol{r}(x) = \left[\left.\frac{\mathrm{d}f}{\mathrm{d}\eta}\right|_{\eta=\eta_{\max}}\right]^2 + \left[\theta|_{\eta=\eta_{\max}} - \theta_\infty\right]^2 + \left[\left.\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\right|_{\eta=\eta_{\max}}\right]^2 + \left[\left.\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right|_{\eta=\eta_{\max}}\right]^2 \tag{18}$$

where

$$x = (x_1, x_2)^T, \qquad x_1 = \left.\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\right|_{\eta=0}, \qquad x_2 = \left.\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right|_{\eta=0}$$

where $\boldsymbol{r}$ is a vector of individual residual functions of dimension 4, where each component is the term inside each pair of square brackets of Eq. (18). F is in the form of the sum of squares of four terms, out of which the first two are the deviation between the calculated values and the actual boundary conditions (3b) and (3d), and the last two terms ensure that f', θ approach asymptotically the values 0 and θ_∞, at $\eta = \eta_{\max}$, when $F = 0$. The objective is to minimize the function F so that the boundary conditions at $\eta = \eta_{\max}$ are satisfied in a least-squares sense. The resulting nonlinear least-squares problem is solved using a modified Levenberg–Marquardt (LM) technique in conjunction with a trust region approach.[9] The LM method is a variant of the Gauss–Newton method for the solution of nonlinear least-squares problems; the latter has been used in pure natural convection problems earlier by Nachtsheim and Swigert.[10] The modified LM method based on the trust region strategy has an excellent convergence rate and is fast because it needs fewer function evaluations. The iterative procedure is continued until the sum of the residual functions is within 1E-6. This determines the correct boundary conditions $\mathrm{d}^2 f/\mathrm{d}\eta^2$, $\mathrm{d}\theta/\mathrm{d}\eta$ at $\eta = 0$. The updated temperature profile evaluated using the corrected boundary conditions is now taken as the new approximation, the radiative divergence term is again calculated, and the scheme is continued until the maximum norm of the relative temperature differences at a number of grid points in the η variable between two successive iterations is below 1E-4. The numerical procedure is then continued for the subsequent stations ξ_{k+1}, ξ_{k+2}, etc.

For the numerical integration of the ordinary differential equations, a variable-order, variable-step Adams–Moulton method with automatic error control is used.[11] The order and the step size are chosen such that the maximum norm of the relative local truncation error is within 1.0E-5. Nonuniform step sizes starting from $\Delta\xi = 0.001$ and increased by a factor of 1.1 at each step are used in the ξ direction, and $\eta_{\max}$ is taken to be 8.0. Forty-eight ($=2m$) directions are used for the radiation intensity calculations in Eqs. (8) and (9). These values are found to be sufficient to give satisfactory grid-independent solutions.

Results and Discussion

Dimensionless temperature profiles for various values of ξ are shown in Fig. 1. Temperature increases with ξ. The results of pure natural convection, i.e., $\xi = 0$, compare very well with the data of Ostrach[12] as presented in Burmeister.[13] The DOM algorithm for the radiation analysis has been validated separately by comparing the results of the present analysis for the radiative flux, between two opposing infinite black parallel plates, against the exact results presented in Modest.[5] The deviation in results is less than 0.1%, lending support to the accuracy of the radiative transfer analysis. Comparison of the present results with that of Cheng and Ozisik[3] for the case of isotropic scattering is shown in Fig. 1.

To investigate the effect of anisotropic radiation scattering in the total heat transfer, we consider a linear anisotropic scattering (LAS)

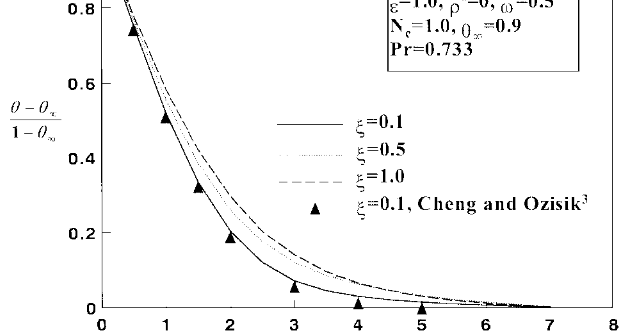

Fig. 1 Dimensionless temperature.

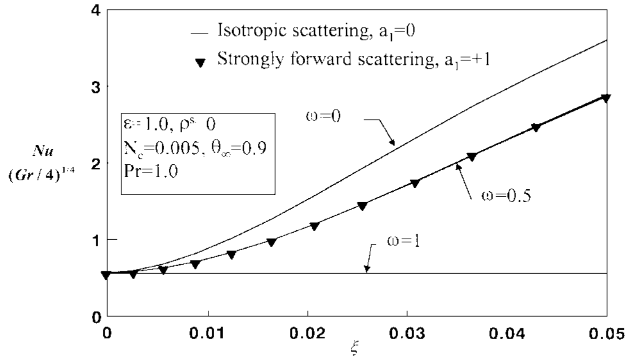

Fig. 2 Effect of scattering phase functions and albedo.

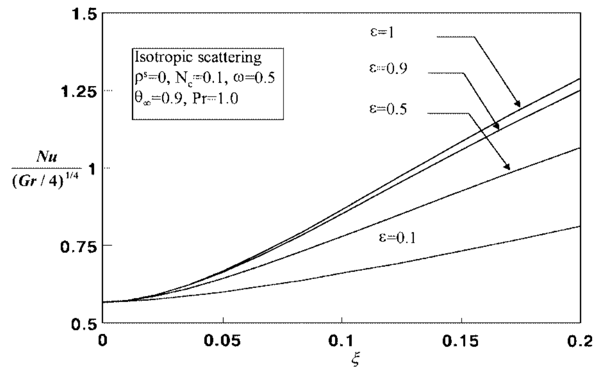

Fig. 3 Effect of emittance.

model. The single-scattering phase function for the LAS model is given as

$$p(\mu_p) = 1 + a_1\mu_p$$

where μ_p is the cosine of the angle between the in-scattering and out-scattering directions: $a_1 = 0$ represents isotropic scattering, $a_1 = +1$ indicates strong forward scattering, whereas $a_1 = -1$ corresponds strong backward scattering. Forward scattering enhances the radiative transfer in the direction from the plate to fluid, whereas backward scattering retards the transfer. The slab phase function $p(\mu, \mu')$ required in Eqs. (8) and (9) can be easily obtained from $p(\mu_p)$[5].

The variation of $Nu/(Gr/4)^{1/4}$ with the dimensionless axial distance ξ is shown in Fig. 2. $Nu/(Gr/4)^{1/4}$ is seen to increase with ξ. The effect of degree of anisotropy a_1 is also presented in Fig. 2. For $a_1 = +1$, i.e., for strong forward scattering, the increase in $Nu/(Gr/4)^{1/4}$ over isotropic scattering is insignificant even at the low value of N_c, where radiation dominates over conduction. Strong backscattering ($a_1 = -1$) also affects only insignificant reduction in $Nu/(Gr/4)^{1/4}$. Increasing in scattering albedo decreases the heat flux as observed from Fig. 2. $\omega = 1$ yields the least heat flux because of the total decoupling of convective and radiative heat fluxes for this case. Figure 3 depicts the variation of heat flux with emittance. At higher emittances and ambient temperatures $Nu/(Gr/4)^{1/4}$ increases, and this is obvious because of larger radiation interactions.

Conclusion

Degree of radiation scattering anisotropy does not appear to affect the total heat flux from the plate to the fluid.

References

[1]Cess, R. D., "The Interaction of Thermal Radiation with Free Convection Heat Transfer," *International Journal of Heat Mass Transfer*, Vol. 9, No. 11, 1966, pp. 1269–1277.

[2]Arpaci, V. S., "Effect of Thermal Radiation on the Laminar Free Convection from a Heated Vertical Plate," *International Journal of Heat Mass Transfer*, Vol. 11, April 1968, pp. 871–881.

[3]Cheng, E. H., and Ozisik, M. N., "Radiation with Free Convection in an Absorbing, Emitting, and Scattering Medium," *International Journal of Heat Mass Transfer*, Vol. 15, No. 6, 1972, pp. 1243–1252.

[4]Ozisik, M. N., *Radiative Transfer and Interactions with Conduction and Convection*, Wiley, New York, 1973, pp. 249–312.

[5]Modest, M. F., *Radiative Heat Transfer*, McGraw–Hill, New York, 1993, pp. 312–315, 450–461, 541–571.

[6]Cheng, P., and Lui, H. C., "Planar Radiating Flow by the Method of Discrete Ordinates," *International Journal of Heat Mass Transfer*, Vol. 13, Oct. 1970, pp. 1793–1798.

[7]Love, T. J., and Grosh, R. J., "Radiative Heat Transfer in Absorbing, Emitting, and Scattering Media," *Journal of Heat Transfer*, Vol. 87, Series C, No. 2, 1965, pp. 161–166.

[8]Auer, L., "Acceleration of Convergence," *Numerical Radiative Transfer*, edited by W. Kalkofen, Cambridge Univ. Press, Cambridge, England, UK, 1987, pp. 101–109.

[9]Dennis, J. E., Jr., and Schnabel, R. B., *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice–Hall, Englewood Cliffs, NJ, 1983.

[10]Nachtsheim, P. R., and Swigert, "Satisfaction of Asymptotic Boundary Conditions in Numerical Solution of System of Nonlinear Equations of Boundary-Layer Type," NASA TN D-3004, 1965.

[11]Gear, C. W., *Numerical Initial Value Problems in Ordinary Differential Equations*, Prentice–Hall, Englewood Cliffs, New Jersey, 1971, pp. 136–168.

[12]Ostrach, S., "An Analysis of Laminar Free-Convection Flow and Heat Transfer About a Flat Plate Parallel to the Direction of the Generating Body Force," NACA TR-1111, 1953.

[13]Burmeister, L. C., *Convective Heat Transfer*, 2nd ed., Wiley, New York, 1993, p. 388.

Combined Convective and Radiative Heat Transfer in Turbulent Tube Flow

C. K. Krishnaprakas* and K. Badari Narayana†
ISRO Satellite Centre, Bangalore 560 017, India
and
Pradip Dutta‡
Indian Institute of Science, Bangalore 560 012, India

Nomenclature

d	= diameter of the tube, $2r_0$, m
f	= Darcy–Weissbach friction factor
g	= incident radiation function
h	= heat transfer coefficient, $q/(T_w - T_b)$, W/m^2-K
$I(\eta, \hat{s})$	= radiation intensity at distance η in the direction $\hat{s}$, W/m^2-sr
$I_b(T)$	= blackbody radiation intensity, W/m^2-sr
k	= thermal conductivity of the medium, W/m-K
N_c	= conduction-radiation number, $k\beta/4\sigma T_w^3$
Nu	= local Nusselt number, hd/k
$\hat{n}$	= normal vector to the surface
Pr	= Prandtl number, ν/α
Pr_t	= turbulent Prandtl number, $\varepsilon_m/\varepsilon_h$
$p(\hat{s}, \hat{s}')$	= scattering phase function
$p(\mu_p)$	= single-scattering phase function
q	= total heat flux by convection and radiation, W/m^2
Re	= Reynolds number, $u_b d/\nu$
r	= radial distance, m
r_0	= radius of the tube, m

Received 26 June 1998; revision received 20 October 1998; accepted for publication 8 March 1999.

*Engineer; currently Research Scholar, Indian Institute of Science, Bangalore 560 012, India.

†Head, Thermal Design Section, Thermal Systems Group.

‡Assistant Professor, Mechanical Engineering Department.